

7 January 1964

UNPUBLISHED PRELIMINARY DATA

19p
Honeywell

N64-25569

CODE-1

Cap 30

NASA CR 56831

Down town
ROCKET BOOSTER CONTROL

SECTION 2

CONTROL OF PLANTS WHOSE
REPRESENTATION CONTAINS DERIVATIVES

OF THE CONTROL VARIABLE

NASA Contract NASw-563

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MILITARY PRODUCTS GROUP RESEARCH DEPARTMENT

7 January 1964

ROCKET BOOSTER CONTROL

SECTION 2

CONTROL OF PLANTS WHOSE
REPRESENTATION CONTAINS DERIVATIVES
OF THE CONTROL VARIABLE

NASA Contract NASw-563

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FOREWORD

This document is one of sixteen sections that comprise the final report prepared by the Minneapolis-Honeywell Regulator Company for the National Aeronautics and Space Administration under contract NASw-563. The report is issued in the following sixteen sections to facilitate updating as progress warrants:

- 1541-TR 1 Summary
- 1541-TR 2 Control of Plants Whose Representation Contains Derivatives of the Control Variable
- 1541-TR 3 Modes of Finite Response Time Control
- 1541-TR 4 A Sufficient Condition in Optimal Control
- 1541-TR 5 Time Optimal Control of Linear Recurrence Systems
- 1541-TR 6 Time-Optimal Bounded Phase Coordinate Control of Linear Recurrence Systems
- 1541-TR 7 Penalty Functions and Bounded Phase Coordinate Control
- 1541-TR 8 Linear Programming and Bounded Phase Coordinate Control
- 1541-TR 9 Time Optimal Control with Amplitude and Rate Limited Controls
- 1541-TR 10 A Concise Formulation of a Bounded Phase Coordinate Control Problem as a Problem in the Calculus of Variations
- 1541-TR 11 A Note on System Truncation
- 1541-TR 12 State Determination for a Flexible Vehicle Without a Mode Shape Requirement
- 1541-TR 13 An Application of the Quadratic Penalty Function Criterion to the Determination of a Linear Control for a Flexible Vehicle
- 1541-TR 14 Minimum Disturbance Effects Control of Linear Systems with Linear Controllers
- 1541-TR 15 An Alternate Derivation and Interpretation of the Drift-Minimum Principle
- 1541-TR 16 A Minimax Control for a Plant Subjected to a Known Load Disturbance

Section 1 (1541-TR 1) provides the motivation for the study efforts and objectively discusses the significance of the results obtained. The results of inconclusive and/or unsuccessful investigations are presented. Linear programming is reviewed in detail adequate for sections 6, 8, and 16.

It is shown in section 2 that the purely formal procedure for synthesizing an optimum bang-bang controller for a plant whose representation contains derivatives of the control variable yields a correct result.

In section 3 it is shown that the problem of controlling m components ($1 < m \leq n$), of the state vector for an n -th order linear constant coefficient plant, to zero in finite time can be reformulated as a problem of controlling a single component.

Section 4 shows Pontriagin's Maximum Principle is often a sufficient condition for optimal control of linear plants.

Section 5 develops an algorithm for computing the time optimal control functions for plants represented by linear recurrence equations. Steering may be to convex target sets defined by quadratic forms.

In section 6 it is shown that linear inequality phase constraints can be transformed into similar constraints on the control variables. Methods for finding controls are discussed.

Existence of and approximations to optimal bounded phase coordinate controls by use of penalty functions are discussed in section 7.

In section 8 a maximum principle is proven for time-optimal control with bounded phase constraints. An existence theorem is proven. The problem solution is reduced to linear programming.

A backing-out-of-the-origin procedure for obtaining trajectories for time-optimal control with amplitude and rate limited control variables is presented in section 9.

Section 10 presents a reformulation of a time-optimal bounded phase coordinate problem into a standard calculus of variations problem.

A mathematical method for assessing the approximation of a system by a lower order representation is presented in section 11.

Section 12 presents a method for determination of the state of a flexible vehicle that does not require mode shape information.

The quadratic penalty function criterion is applied in section 13 to develop a linear control law for a flexible rocket booster.

In section 14 a method for feedback control synthesis for minimum load disturbance effects is derived. Examples are presented.

Section 15 shows that a linear fixed gain controller for a linear constant coefficient plant may yield a certain type of invariance to disturbances. Conditions for obtaining such invariance are derived using the concept of complete controllability. The drift minimum condition is obtained as a specific example.

In section 16 linear programming is used to determine a control function that minimizes the effects of a known load disturbance.

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CONTROL OF SYSTEMS CONTAINING
DERIVATIVES OF THE CONTROL
VARIABLE*

by Wayne Schmaedeke†

ABSTRACT

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Problems of control of plants modeled by differential equations containing derivatives of the control (or forcing) functions are discussed. These right side plant dynamics in conjunction with relay type jumps in the forcing functions require an analysis or synthesis method to accommodate derivatives of step functions. Previously this problem has been avoided formally by transforming to a special set of coordinates in which only zero order forcing terms appear. This paper develops a mathematical model for equations with derivatives of the control functions and establishes conditions under which the formal transformation referred to above can be rigorously applied.

Author

MATHEMATICAL MODEL OF A CONTROL SYSTEM CONTAINING
DERIVATIVES OF THE CONTROL VARIABLES

In the following, \mathcal{D} will be a domain in the (t,x) -space, $f(t,x)$ will be an n -vector function defined on \mathcal{D} and $u(t)$ will be an r -vector each of whose components are of bounded variation and continuous from the right on an interval I_1 . The function

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$g(t)$ will be a continuous $n \times r$ matrix defined on I_1 and (t_0, x_0) will be a point in \mathcal{D} with t_0 also in I_1 .

In a control problem, one is given a differential equation

$$\frac{dx}{dt} = f(t, x, u) + g(t) \frac{du}{dt} \quad (8)$$

involving f , g , u , and x . The operations of differentiation are to be understood in the sense of distribution derivatives and the equation will be called a measure differential equation because the distribution derivative of a function of bounded variation can always be identified with a measure. The problem of control is to select the r -vector $u(t)$ on an interval of time $[t_0, t_1]$ so that the solution (response) of (8) will initiate at a prescribed point x_0 at the time t_0 and behave in a prescribed manner on the interval $[t_0, t_1]$. For example, it may be desired to steer the response from x_0 to a continuously moving target set $G(t) \subset R^n$.

It will be convenient to change the notation in (8) to conform with the previous remarks and to write the equation as

$$Dx = f(t, x, u) + g(t)Du, \quad x(t_0) = x_0. \quad (9)$$

Here the notation Dx means the distribution derivative of the function x .

DEFINITION 1. A solution $x(t)$ of (9) is a real bounded variation n -vector $x(t)$ together with an interval I containing the given initial time t_0 such that $x(t)$ is continuous from the right on I and

- (i) $(t, x(t)) \in \mathcal{D}$ for $t \in I$
- (ii) $x(t_0) = x_0$
- (iii) the distribution derivative of $x(t)$ on I is $f(t, x) + g(t)Du$.

Next, the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s), u(s))ds + \int_{t_0}^t g(s)du(s). \quad (\mathcal{Q})$$

will be considered.

DEFINITION 2. A solution $x(t)$ of (\mathcal{Q}) is a real bounded variation n -vector $x(t)$ together with an interval I such that

- (i) $(t, x(t)) \in \mathcal{D}$ for $t \in I$
- (ii) $x(t)$ satisfies the integral equation

REMARK: A solution $x(t)$ of (\mathcal{Q}) is necessarily continuous from the right. Also, $x(t)$ has discontinuities where u does.

THEOREM 1. A solution $x(t)$ of (\mathcal{Q}) is a solution $x(t)$ of (\mathcal{M}) and conversely.

A proof of Theorem 1 is given in reference 1. Also included in that reference are a number of theorems relating to existence and uniqueness of solutions to (\mathcal{M}) , both locally and globally.

For convenience in discussing the control problem for an equation of the form (\mathcal{M}) , the following assumptions regarding the coefficients are made:

- (i) $f(t, x, u), \frac{\partial f}{\partial x}(t, x, u)$ are real continuous functions in $R^1 \times R^n \times \Omega$ where R^n is the real n -dimensional number space and Ω is a non-empty compact subset of R^n .

- (ii) $g(t)$ is a continuous $n \times r$ -matrix on R^1
- (iii) The functions $u(t)$ are of bounded variation and continuous from the right on appropriate time intervals such that the graph of $u(t)$ lies in Ω .

For each function $u(t)$ as in (iii), but defined on a finite interval $[t_0, t_1]$, the measure differential equation (m) has a unique bounded variation solution (called a response) on $[t_0, t_1]$ (or a subinterval) through a prescribed initial point (t_0, x_0) . This is a result of the existence and uniqueness theorems of Chapter I of reference 1. The representation of a response is, of course, the unique bounded variation solution of the integral equation

$$x(t) = x_0 + \int_{t_0}^t f(s, x(s), u(s)) ds + \int_{t_0}^t g(s) du(s). \quad (4)$$

DEFINITION 3. A control for the system (m) where a non-empty compact set Ω contained in R^r and an initial point x_0 in R^n have been prescribed, is a vector valued function $u(t)$ of bounded variation and continuous from the right on a finite interval $[t_0, t_1]$ with its graph in Ω such that its response $x(t)$ with $x(t_0) = x_0$ is also defined in R^n on $[t_0, t_1]$.

DEFINITION 4. For a given real valued continuous function $f^0(t, x, u)$ defined on $R^1 \times R^n \times \Omega$, the cost functional $C(u)$ of a control $u(t)$ on $[t_0, t_1]$ with response $x(t)$ is defined by

$$C(u) = \int_{t_0}^{t_1} f^0(s, x(s), u(s)) ds.$$

If $f^0(t, x, u) \equiv 1$ then $C(u) = t_1 - t_0$, the time duration over which the control is exerted.

DEFINITION 5. Given the control problem for

- a) $Dx = f(t, x, u) + g(t)Du$ with f , g , and u as previously described; the following data is also given:
- b) a non-empty compact restraint set $\Omega \subset R^r$ (containing the graphs on the controls)
- c) $x_0 \in R^n$, the initial point
- d) $G(t) \subset R^n$ on $[\tau_0, \tau_1]$ the continuously moving target set
- e) $C(u) = \int_{t_0}^{t_1} f^0(s, x(s), u(s)) ds$, the cost functional.

For a given number $E > 0$, the set $\Delta = \Delta(f(t, x, u), g(t), \Omega, x_0, G(t), E)$ is defined as the set of all controls $u(t)$ in Ω with $u(t)$ of bounded variation and right continuous on subintervals $[t_0, t_1] \subset [\tau_0, \tau_1]$ such that the total variation of each function $u(t)$ on its interval $[t_0, t_1]$ is less than or equal to E , and with responses $x(t)$ such that $x(t_0) = x_0$ and $x(t_1) \in G(t_1)$. This set Δ is called the set of admissible controls.

DEFINITION 6. A control $u^*(t)$ in Δ is called optimal in case

$$C(u^*) \leq C(u)$$

for every $u(t)$ in Δ .

REMARK: The hypothesis concerning the uniform bounded total variation of the admissible controls is concerned with the fact that in a large class of problems the total variation is a mathematical manifestation of the motion of some process. It is those processes which contain devices capable of sustaining only a bounded amount of movement, regardless of the admissible control that is applied, to which the following theorem pertains.

THEOREM 2. Given the control problem described in Definition 5 with the further restriction that there exist a non-decreasing function $h(t)$ continuous from the right such that all $u(t)$ in Δ satisfy the inequalities

$$|u(\beta) - u(\alpha)| \leq h(\beta) - h(\alpha)$$

for every subinterval $[\alpha, \beta]$ of the interval $[t_0, t_0 + \delta]$ for some appropriate $\delta > 0$, however small.

It will be assumed that the set Δ is such that

- A) Δ is not empty
- B) There exists a real bound $B < \infty$ such that for all responses $x(t)$ corresponding to controls in Δ we have $|x(t)| \leq B$.

CONCLUSION: Then there exists an optimal control in Δ .

REMARK: It is assumed that x_0 is not in the target $G(t_0)$, then the inequalities between u and h guarantee that $x(t)$ lies outside $G(t)$ for all responses and all t sufficiently near t_0 . If the functions $u(t)$ in Δ satisfy a uniform Lipschitz condition for all t in a neighborhood of t_0 then the function $h(t)$ may be taken to be a multiple of t .

The proof of Theorem 2 may be found in reference 1.

A STUDY OF A CERTAIN LINEAR EQUATION

The linear ordinary differential equation

$$\ddot{x} + a_1 \dot{x} + a_0 x = b_1 \dot{u} + b_2 u. \quad (\mathcal{E})$$

will be considered. This equation might be obtained, for example, in the analysis of the control of aircraft. (Reference 2, where the equations of longitudinal-symmetric motion of a rigid aircraft

are derived, contains the details). In particular, the small amplitude pitch motions of many aircraft can be summarized approximately by solutions of the equations

$$\dot{\alpha} - k_1 \alpha - \dot{\theta} = k_2 \delta \quad (\text{Flight Path equation}) \quad (1)$$

$$\ddot{\theta} - k_3 \dot{\theta} - k_4 \alpha - k_5 \dot{\alpha} = k_6 \delta \quad (\text{Pitch equation}) \quad (2)$$

Here the k_j are constants while α corresponds to the aircraft fuselage reference angle of attack, θ corresponds to pitch attitude change, and δ denotes elevator deflection. It will be convenient to define $x = \dot{\theta}$ and to eliminate α and $\dot{\alpha}$ from (1), (2) and the equation obtained from differentiating (2). The result is

$$\ddot{x} - (k_1 + k_3 + k_5)\dot{x} + (k_1 k_3 - k_4)x = (k_6 + k_2 k_5)\dot{\delta} + (k_2 k_4 - k_1 k_6)\delta \quad (3)$$

The equation (3) is precisely of the form (E). It is a problem of considerable importance in engineering to treat this equation by allowing the control variable $u(t)$ to be of the relay type, i.e., to have discontinuities of the first kind. The presence of a derivative of $u(t)$ prevents existing theories from being applied and the problem is avoided formally by transforming to a set of coordinates in which the derivatives of the control variable do not appear, (reference 3, p. 191 supplies the details).

To be more specific now, the equation (E) will be considered where a_1, a_0, b_1, b_2 are constants and where $x(t_0) = x_0, \dot{x}(t_0) = \dot{x}_0, |u(t)| \leq \alpha$ for $\alpha > 0$, and $a_1, a_0 > 0, b_1 \neq 0, b_2 \neq 0$. The target will be any compact set containing the origin of the (x, \dot{x}) -plane and the cost functional will be given as

$$C(u) = \int_{t_0}^{t_1} f^0(x(s), \dot{x}(s), u(s)) ds.$$

At this point (ξ) will be reduced to a linear system free from derivatives of the control variable, (reference 3 p. 191 supplies the details). To this end, x is defined by

$$x = \hat{x}_1 + G_0(t)u$$

for as yet undetermined, $G_0(t)$ and \hat{x}_2 is defined by

$$\dot{\hat{x}}_1 = \hat{x}_2 + G_1(t)u$$

for undetermined $G_1(t)$. Then it is clear that $\dot{\hat{x}}_2$ is given by

$$\dot{\hat{x}}_2 = -a_0\hat{x}_1 - a_1\hat{x}_2 + G_2(t)u$$

where G_0 , G_1 , and G_2 are determined by eliminating \hat{x}_1 and \hat{x}_2 from the above and requiring that the resultant differential equation agree with (ξ). The result is

$$G_0(t) = 0$$

$$G_1(t) = b_1$$

$$G_2(t) = b_2 - a_1b_1.$$

The system

$$\dot{\hat{x}}_1 = \hat{x}_2 + b_1u$$

(9)

$$\dot{\hat{x}}_2 = -a_0\hat{x}_1 - a_1\hat{x}_2 + (b_2 - a_1b_1)u$$

is obtained with initial conditions

$$\hat{x}_1(t_0) = x_0$$

$$\hat{x}_2(t_0) = C_0 - b_1u(t_0).$$

It is observed that the initial point (x_0, C_0) has been transformed to a line segment in the (\hat{x}_1, \hat{x}_2) -plane and that the target G would be similarly "enlarged". For simplicity, it is assumed that the origin of the (\hat{x}_1, \hat{x}_2) -plane is in the interior of $G(\hat{x}_1, \hat{x}_2)$. The cost functional becomes

$$C_0(u) = \int_{t_0}^{t_1} f^0(\hat{x}_1(s), \hat{x}_2(s) + b_1u(s), u(s))ds.$$

The ordinary system (8) does not contain derivatives of the control variables and can therefore be studied using the conventional theory.

Next, (8) is written as a linear system by making no attempt to eliminate the derivatives of the control variable, i.e., by proceeding in a natural way by defining

$$x_1 = x$$

the following system is obtained:

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a_0 x_1 - a_1 x_2 + b_1 \dot{u} + b_2 u.\end{aligned}\tag{8'}$$

At this point, the class of admissible controls is taken to be the functions $u(t)$ of uniform bounded variation, with $v(u, [t_0, t_1]) \leq M$, such that $|u| \leq \alpha$.¹ Then (8') is written as a measure differential system

$$\begin{aligned}Dx_1 &= x_2 \\ Dx_2 &= -a_0 x_1 - a_1 x_2 + b_1 Du + b_2 u.\end{aligned}\tag{9}$$

The initial conditions in this instance are

$$\begin{aligned}x_1(t_0) &= x_0 \\ x_2(t_0) &= c_0\end{aligned}$$

and the cost functional is

¹ $v(u, [t_0, t_1])$ means the total variation of the vector $u(t)$ on the interval $[t_0, t_1]$, i.e., each component of $u(t)$ is of uniform bounded total variation.

$$C_m(u) = \int_{t_0}^{t_1} f^0(x_1(s), x_2(s), u(s)) ds.$$

The target is assumed to have been defined by $G(x, \dot{x}) \leq 0$ in which case the target becomes the set in the (x_1, x_2) -plane defined by $G(x_1, x_2) \leq 0$.

It is shown in reference 1 that the set Δ_m for this problem is not empty.

Now the responses to a linear system are easily shown to be uniformly bounded and the set Δ of admissible controls with variations limited by some very large constant E which transfer $X_0 = (x_1(t_0), x_2(t_0))$ to the origin is clearly not empty since the domain of controllability is the entire (x_1, x_2) -plane. Hence there exists an optimal control $u^*(t)$ in Δ which transfers the point $X_0 = (x_0, c_0)$ to the target $G(x_1, x_2) \leq 0$ because the target was assumed to contain the origin of R^2 . The question resolved below is under what conditions will $u^*(t)$ also be the optimal control for the system (\mathcal{O}) when that system is studied using the conventional methods?

A connection is first established between the components of the two vector solutions $X(t) = (x_1(t), x_2(t))'$ and $\hat{X}(t) = (\hat{x}_1(t), \hat{x}_2(t))'$ of (m) and (\mathcal{O}) respectively.

THEOREM 3. If $u(t)$ is any function of bounded variation, then the first components of $X(t)$ and $\hat{X}(t)$ are identical and the second components are related by

$$x_2(t) - \hat{x}_2(t) = b_1 u(t).$$

PROOF: It is observed that the systems (\mathcal{O}) and (m) have the same homogeneous part and that the fundamental solution matrix

is e^{At} where

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}.$$

Hence, forming the difference $X(t) - \hat{X}(t)$ by using the variation of parameters representations for the solutions of each, there results

$$\begin{aligned} X(t) - \hat{X}(t) &= e^{A(t-t_0)}(X_0 - \hat{X}_0) + \int_{t_0}^t e^{A(t-s)} \begin{bmatrix} -b_1 \\ a_1 b_1 \end{bmatrix} u(s) ds + \\ &+ \int_{t_0}^t e^{(t-s)A} \begin{bmatrix} 0 \\ b_1 \end{bmatrix} du(s). \end{aligned}$$

Now, by integration by parts:

$$\begin{aligned} \int_{t_0}^t e^{A(t-s)} \begin{bmatrix} 0 \\ b_1 \end{bmatrix} du(s) &= - \int_{t_0}^t \frac{d}{ds} e^{A(t-s)} \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u(s) ds + \\ &+ \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u(t) - e^{A(t-t_0)} \begin{bmatrix} 0 \\ b_1 \end{bmatrix} u_0. \end{aligned}$$

Thus

$$\begin{aligned} X(t) - \hat{X}(t) &= e^{A(t-t_0)} \left[X_0 - \hat{X}_0 - \begin{bmatrix} 0 \\ b_1 u_0 \end{bmatrix} \right] + \begin{bmatrix} 0 \\ b_1 u(t) \end{bmatrix} + \\ &+ \int_{t_0}^t e^{A(t-s)} \left\{ \begin{bmatrix} 0 \\ b_2 \end{bmatrix} + A \begin{bmatrix} 0 \\ b_1 \end{bmatrix} - \begin{bmatrix} b_1 \\ b_2 - a_1 b_1 \end{bmatrix} u(s) \right\} ds, \end{aligned}$$

but $A \begin{bmatrix} 0 \\ b_1 \end{bmatrix} = \begin{bmatrix} b_1 \\ -a_1 b_1 \end{bmatrix}$ and

$$X_0 - \hat{X}_0 - \begin{bmatrix} 0 \\ b_1 u_1 \end{bmatrix} = \begin{bmatrix} x_0 \\ c_0 \end{bmatrix} - \begin{bmatrix} x_0 \\ c_0 - b_1 u_0 \end{bmatrix} - \begin{bmatrix} 0 \\ b_1 u_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

and the above reduced to

$$x(t) - \hat{x}(t) = \begin{bmatrix} 0 \\ b_1 u(t) \end{bmatrix}. \quad \text{QED.}$$

A comparison of the cost functionals $C_\theta(u)$ and $C_m(u)$ when u is a function of bounded variation shows by virtue of Theorem 3 that they are identical.

Returning now to the study of system (θ) , Δ_θ is defined to be the set of all measurable functions $u(t)$ whose graphs lie in the set $\Omega: \{u: |u| \leq \alpha\}$ and whose responses originate on the closed interval $\hat{x}_1(t_0) = x_0, C_0 - \frac{\alpha}{2} \leq \hat{x}_2(t_0) \leq C_0 + \frac{\alpha}{2}$, and which are transferred to the target at time t_1 . It is shown in reference 1 that Δ_θ is not empty.

It is to be noticed that if $u^*(t)$ is substituted in $C_\theta(u)$, there results

$$C_\theta(u^*) \geq C_\theta(u).$$

Next, since the system $\theta)$ is proper (i.e., $\text{rank } [B, AB] = 2$) then the optimal control $u(t)$ will be relay if, for example, u appears linearly in $f^0(x, t)$, the integrand of the cost functional. Being a relay or step function (assuming values $+\alpha$ or $-\alpha$ on finite segments of time) the optimal control u is of bounded variation. Thus the two cost functionals $C_\theta(u)$ and $C_m(u)$ are identical and there results (by definition of u^* as minimizing $C_m(u)$ over all bounded variation controls whose total variation is uniformly bounded)

$$C_m(u^*) \leq C_m(\hat{u}).$$

But also the inequalities

$$C_m(\hat{u}) = C_\theta(\hat{u}) \leq C_\theta(u^*) = C_m(u^*) \leq C_m(\hat{u})$$

imply $C_{\theta}(\hat{u}) = C_{\theta}(u^*)$ and thus $\hat{u}(t)$ is an optimal control for the system (\mathcal{M}) because it has the desired response. Also when transforming back to x and \dot{x} in (ξ) there results

from (\mathcal{M}) :

$$x(t) = x_1(t)$$

$$\dot{x}(t) = x_2(t)$$

from (θ) :

$$x(t) = \hat{x}_1(t)$$

$$\dot{x}(t) = \hat{x}_2(t) + b_1 u(t).$$

But by Theorem 3, these are identical sets of data.

CONCLUSION

The optimal control problem for

$$\ddot{x} + a_1 \dot{x} + a_0 x = b_1 \ddot{u} + b_2 \dot{u} \quad (\xi)$$

can be treated either by the system

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + b_1 u \\ \dot{\hat{x}}_2 &= -a_0 \hat{x}_1 - a_1 \hat{x}_2 + (b_2 - a_1 b_1) u \end{aligned} \quad (\theta)$$

or by the measure system

$$Dx_1 = x_2 \quad (\mathcal{M})$$

$$Dx_2 = -a_0 x_1 - a_1 x_2 + b_1 Du + b_2 u.$$

The system (θ) involves an enlargement of the initial state and target and leads to confusion. However, the theory of optimal control developed above for measure systems (\mathcal{M}) enables a straight forward treatment of the optimal control to be given which agrees with (θ) .

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